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# ONO INVARIANTS OF IMAGINARY QUADRATIC FUNCTION FIELDS

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Abstract. In this paper, we define the Ono invariants of imaginary quadratic function fields and obtained several results concerning the relations between the Ono invariants and the class numbers.

## 1. Introduction

Let $K$ be an imaginary quadratic number field. Let $h_K$ be the class number of $K$. In [3], Sasaki defined a natural number $\rho_K$ associated to $K$, which is called the *Ono invariant of* $K$, and reformulated the Rabinovitch's theorem as $h_K = 1$ if and only if $\rho_K = 1$. He also proved that $h_K \geq \rho_K$ and $h_K = 2$ if and only if $\rho_K = 2$.

The aim of this paper is to define the Ono invariant of imaginary quadratic function field and prove similar results. Let $k = \mathbb{F}_q(t)$ be the rational function field over the finite field $\mathbb{F}_q$ and $\mathbb{A} = \mathbb{F}_q[t]$. Let $\infty$ be the *infinite place* of $k$ associated to $1/t$. Let $K$ be a (geometric) quadratic extension of $k$. We say that $K$ is *real* if $\infty$ splits in $K$ and *imaginary* otherwise. Let $K$ be an imaginary quadratic extension of $k$. Let $\mathcal{O}_K$ be the integral closure of $\mathbb{A}$ in $K$ and $h_K$ be the ideal class number of $\mathcal{O}_K$. In this paper, following Sasaki [3], we define the Ono invariant $\rho_K$ of $K$ and prove several similar results.

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## 2. Odd characteristic case

In this section we assume that $q$ is odd. Let $\gamma$ be a generator of $\mathbb{F}_q^*$. For any $0 \neq N \in \mathbb{A}$, let $sgn(N)$ denote the leading coefficient of $N$. Then any imaginary quadratic extension $K$ of $k$ can be written uniquely as $K = k(\sqrt{D})$, where $D$ is a square free of odd degree if $\infty$ ramifies in $K$ and $D$ is a square free of even degree with $sgn(D) = \gamma$ if $\infty$ is inert in $K$. The integral closure $\mathcal{O}_K$ of $\mathbb{A}$ in $K$ is $\mathcal{O}_K = \mathbb{A}[\sqrt{D}]$.

### 2.1. Ideals

Let $\mathcal{N}$ be the norm map of $K$ into $k$. Any ideal $\mathfrak{a}$ of $\mathcal{O}_K$ can be written as $\mathfrak{a} = [T, R + S\sqrt{D}] := T\mathbb{A} + (R + S\sqrt{D})\mathbb{A}$, where $T$ is the monic polynomial of the smallest degree which is contained in $\mathfrak{a}$ and $S$ is the monic polynomial of the smallest degree such that $R + S\sqrt{D} \in \mathfrak{a}$ for some $R \in \mathbb{A}$. Moreover, we may assume that $\deg R < \deg T$ and $T, R$ are divisible by $S$, and we have $\mathcal{N}\mathfrak{a} = (TS) = TS\mathbb{A}$.

Lemma 2.1. *Let $T, R \in \mathbb{A}$ with $T$ monic. Then the $\mathbb{A}$-module $[T, R + \sqrt{D}]$ becomes an ideal of $\mathcal{O}_K$ if and only if $T|\mathcal{N}(R + \sqrt{D})$. In this case the followings hold:*

(i) *If $T = T_1T_2$, then $[T, R + \sqrt{D}] = [T_1, R + \sqrt{D}][T_2, R + \sqrt{D}]$.*
(ii) *If $0 < \deg T < \deg D$, then $[T, R + \sqrt{D}]$ is not a principal ideal.*

*Proof.* Suppose that $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$. Since $\mathcal{N}(R + \sqrt{D}) = (R + \sqrt{D})(R - \sqrt{D}) \in \mathfrak{a}$, $\mathcal{N}(R + \sqrt{D}) = AT + B(R + \sqrt{D})$ for some $A, B \in \mathbb{A}$. Then $B = 0$ and $AT = \mathcal{N}(R + \sqrt{D})$. Conversely, assume that $\mathcal{N}(R + \sqrt{D})$ is divisible by $T$, say $\mathcal{N}(R + \sqrt{D}) = AT$. Then, for any $\alpha = X + Y\sqrt{D} \in \mathcal{O}_K$, we have $\alpha T = (X - YR)T + TY(R + \sqrt{D}) \in \mathfrak{a}$ and $\alpha(R + \sqrt{D}) = -AYT + (X + YR)(R + \sqrt{D}) \in \mathfrak{a}$. Hence $\mathfrak{a}$ is an ideal of $\mathcal{O}_K$. (i) is obvious. For (ii), we assume that $\mathfrak{a} = [T, R + \sqrt{D}]$ is a principal ideal $(\alpha)$. Then $\alpha = TX + (R + \sqrt{D})Y$ for some $X, Y \in \mathbb{A}$, so $\mathcal{N}\mathfrak{a} = T\mathbb{A} = \mathcal{N}(\alpha)\mathbb{A} = T\big(TX^2 + 2RXY + AY^2\big)\mathbb{A}$, where $TA = R^2 - D$. Thus we get $(TX + RY)^2 = DY^2 + cT$ for some $c \in \mathbb{F}_q$. If $Y = 0$, we have $T^2X^2 = cT$, which is impossible. Suppose $Y \neq 0$. Consider the case that $K$ is a ramified imaginary quadratic extension. Since $\deg T < \deg D$, $\deg(DY^2 + cT) = \deg D + 2\deg Y$ is odd, but $\deg((TX + RY)^2) = 2\deg(TX + RY)$ is even, which is a contradiction. Now, consider the case that $K$ is a inert imaginary quadratic extension. Since $cT = (TX + RY)^2 - DY^2$ and $D$ is of even degree, we have $\mathrm{sgn}(D) = \gamma$, $\mathrm{sgn}(TX + RY)^2 = \gamma^{2m}$ and $\mathrm{sgn}(DY^2) =$

$\gamma^{2n+1} \neq \mathrm{sgn}(TX+RY)^2$ for some $m, n$. Thus $\deg((TX+RY)^2 - DY^2) = max\{\deg(TX+RY)^2, \deg DY^2\} \geq \deg DY^2 = \deg D + 2\deg Y > \deg T$, which contradicts to the fact that $\deg T < \deg D$. $\square$

LEMMA 2.2. *Let $T, R \in \mathbb{A}$ with $T$ monic, $T|\mathcal{N}(R+\sqrt{D})$ and $\deg \mathcal{N}(R+\sqrt{D}) < 2\deg D$. Then the ideal $[T, R+\sqrt{D}]$ is principal if and only if $T = 1$ or $T = \mathcal{N}(R+\sqrt{D})$.*

*Proof.* If $T = 1$, then $[T, R+\sqrt{D}] = [1, \sqrt{D}] = \mathcal{O}_K$, and, if $T = \mathcal{N}(R+\sqrt{D}) = (R+\sqrt{D})(R-\sqrt{D})$, then $[T, R+\sqrt{D}] = (R+\sqrt{D})$. Conversely, let $T' = \mathcal{N}(R+\sqrt{D})/T$, so $TT' = \mathcal{N}(R+\sqrt{D}) = R^2 - D$. Since $\deg(R^2 - D) < 2\deg D$, $\deg T < \deg D$ or $\deg T' < \deg D$. Since $[T, R+\sqrt{D}][T', R+\sqrt{D}] = [TT', R+\sqrt{D}] = (R+\sqrt{D})$, $[T', R+\sqrt{D}]$ is also principal. If $\deg T < \deg D$, since $[T, R+\sqrt{D}]$ is principal, $T = 1$ (by Lemma 2.1 (ii)). If $\deg T' < \deg D$, similarly, $T' = 1$, so $T = \mathcal{N}(R+\sqrt{D})$. $\square$

### 2.2. Ono invariant $\rho_K$

Let $\omega(N)$ be the number of irreducible factors (counting multiplicity) of $N \in \mathbb{A}$. Define the polynomial $f_D(\mathbf{x}) = \mathbf{x}^2 - D$. Then the *Ono invariant $\rho_K$ of $K$* is defined to be

$$\rho_K := \max\big\{\omega(f_D(R)) : \deg R < \deg D\big\}.$$

THEOREM 2.3. *$h_K \geq \rho_K$.*

*Proof.* By definition, $\rho_K = \omega(f_D(R))$ for some $R$ with $\deg R < \deg D - 1$. Note that $f_D(R) = R^2 - D = \mathcal{N}(R+\sqrt{D})$ with $\deg N(R+\sqrt{D}) < 2\deg D$. We may assume that $n = \omega(f_D(R)) \geq 1$. Let $f_D(R) = P_1 \cdots P_n$ be the (monic) irreducible factorization of $f_D(R)$. Write $\mathfrak{a}_i = [P_i, R+\sqrt{D}]$, which is an ideal of $\mathcal{O}_K$ for $1 \leq i \leq n$. We claim that the ideal classes $(\mathfrak{a}_1), (\mathfrak{a}_1\mathfrak{a}_2), \ldots, (\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_n)$ are mutually distinct. Then $h_K \geq n = \rho_K$ follows immediately. It remains to prove the claim. Assume that $(\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_i) = (\mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_j)$ for some $i < j$. Then $(\mathfrak{a}_{i+1}\cdots\mathfrak{a}_j) = ([P_{i+1}\cdots P_j, R+\sqrt{D}]) = 1$, so $[P_{i+1}\cdots P_j, R+\sqrt{D}]$ is a principal ideal. But, taking $T = P_{i+1}\cdots P_j | f_D(R) = N(R+\sqrt{D})$, we have that $[T, R+\sqrt{D}] = [P_{i+1}\cdots P_j, R+\sqrt{D}]$ is a principal ideal with $T \neq 1, N(R+\sqrt{D})$, which contradicts to Lemma 2.2. $\square$

LEMMA 2.4. **(Minkowski's Lemma)** *Any ideal class of $K$ contains an integral ideal $\mathfrak{a}$ such that $N\mathfrak{a} \leq g_K$, where $g_K$ is the genus of $K$.*

Note that the genus $g_K$ of $K = k(\sqrt{D})$ is given by $g_K = \frac{\deg D-1}{2}$ if $\infty$ ramifies in $K$ and $g_K = \frac{\deg D-2}{2}$ if $\infty$ is inert in $K$.

THEOREM 2.5. $\rho_K = 1$ *if and only if* $h_K = 1$.

*Proof.* By Theorem 2.3, it is clear that if $h_K = 1$, then $\rho_K = 1$. Now we will show that if $\rho_K = 1$, then $h_K = 1$. Suppose that $h_K \geq 2$. Let $\mathfrak{p}$ be a non-principal ideal having the smallest $\deg N\mathfrak{p}$. It can be shown that $\mathfrak{p}$ is a prime ideal and $N\mathfrak{p} = P$ is a monic irreducible. By Lemma 2.4, we have $\deg P \leq \frac{\deg D-1}{2}$. Write $\mathfrak{p} = [P, R + \sqrt{D}]$ for some $R \in \mathbb{A}$ with $\deg R < \deg P$ and $P|\mathcal{N}(R + \sqrt{D})$. Then $\deg N(R + \sqrt{D}) = \deg(R^2 - D) < 2\deg D$. Since $\rho_K = 1$, $f_D(R) = N(R+\sqrt{D}) = R^2-D$ must be an irreducible. Hence, $P = \mathcal{N}(R + \sqrt{D})$ and $\mathfrak{p} = [P, R + \sqrt{D}] = (R + \sqrt{D})$ is principal, which is a contradiction. $\square$

THEOREM 2.6. $\rho_K = 2$ *if and only if* $h_K = 2$.

*Proof.* If $h_K = 2$, then $\rho_K \leq h_K = 2$, so $\rho_K = 1$ or $2$. But, by Theorem 2.5, we have $\rho_K = 2$. Conversely, assume that $\rho_K = 2$. Then $h_K \geq \rho_K = 2$. Let $\mathfrak{p}, \mathfrak{q}$ be non-principal prime ideals with $N\mathfrak{p} = P, N\mathfrak{q} = Q$ and $\deg P, \deg Q \leq \frac{\deg D-1}{2}$. Then $P, Q$ are monic irreducible ones. We can show that $\mathfrak{p}\mathfrak{q}$ is principal.

Now we will show that every ideal class is of order $\leq 2$. Let $\mathfrak{p}$ be an ideal such that $\mathfrak{p}^2$ is non-principal and has the smallest degree. Then $\mathfrak{p}$ is a prime ideal and $N\mathfrak{p} = P$ is a monic irreducible with $\deg P \leq \frac{\deg D-1}{2}$. By the above argument, $\mathfrak{p}^2$ is principal. Thus, every ideal class of $K$ is of order $\leq 2$, so the ideal class group of $K$ is an elementary abelian 2-group. Now suppose $h_K \geq 4$. Let $\mathfrak{p}$ be a non-principal ideal having the smallest degree and $\mathfrak{q}$ a non-principal ideal such that $\mathfrak{q}$ is not equivalent to $\mathfrak{p}$ and has the smallest degree. As above, $\mathfrak{p}\mathfrak{q}$ is principal. Since $\mathfrak{p}^2, \mathfrak{q}^2$ are principal, $\mathfrak{q}$ is equivalent to $\mathfrak{p}$, which is a contradiction. Therefore $h_K = 2$. $\square$

## 3. Even characteristic case

In this section we assume that $q$ is even. Any quadratic extension $K$ of $k$ can be written as $K = k(\alpha)$, where $\alpha$ is a root of $\mathbf{x}^2 + \mathbf{x} = \frac{D_1}{D_2}$ with $D_1 \in \mathbb{A}, D_2 \in \mathbb{A}^+, \gcd(D_1, D_2) = 1$ ([1, 2]). Let $\wp(\mathbf{x}) = \mathbf{x}^2 + \mathbf{x}$ be the Artin-Schreier operator. We say that $u = \frac{D_1}{D_2}$ is *normalized* if it satisfies the following conditions: (i) if $D_2 = \prod_{i=1}^{s} P_i^{e_i}$, then each $e_i$ is odd, (ii) if $\deg D_1 > \deg D_2$, then $\deg D_1 - \deg D_2$ is odd, (iii) if $\deg D_1 = \deg D_2$, then $sgn(D_1) \notin \wp(\mathbb{F}_q)$. We say that the extension $K/k$ is real, inert imaginary or ramified imaginary according as $\infty$ splits, inert or ramifies in $K$. Then, the extension $K/k$ is real, inert imaginary or ramified imaginary according as $\deg D_1 < \deg D_2$, $\deg D_1 = \deg D_2$ or $\deg D_1 > \deg D_2$.

Let $G = G(D_1/D_2) = \prod_{i=1}^{s} P_i^{(e_i+1)/2}$, $Q = Q(D_1/D_2) = \prod_{i=1}^{s} P_i$ and $y = G\alpha$. Then $\mathcal{O}_K = \mathbb{A}[y] = \mathbb{A} + y\mathbb{A} = [1, y]$ is the integral closure of $\mathbb{A}$ in $K$. Note that $y$ is a zero of $\mathbf{x}^2 + G\mathbf{x} + D_1Q = 0$. In addition, a prime $P$ of $\mathbb{A}$ is ramified in $K$ if and only if $P$ divides $G$, and $g(K) = \deg G - 1$ by [4, Prop. III.7.8.(d)].

### 3.1. Ideals

For any integral ideal $\mathfrak{a}$ of $\mathcal{O}_K$, we have $\mathfrak{a} = [T, R + Sy] = \mathbb{A} \cdot T + \mathbb{A} \cdot (R + Sy)$, where $T$ is the monic polynomial of the smallest degree which is contained in $\mathfrak{a}$ and $S$ is the monic polynomial of the smallest degree such that $R + SG\alpha \in \mathfrak{a}$ for some $R \in \mathbb{A}$. Moreover, we may assume that $\deg R < \deg T$ and $T, R$ are divisible by $S$, and $\mathcal{N}\mathfrak{a} = (TS) = TS\mathbb{A}$.

The following two lemmas are even characteristic version of Lemma 2.1 and Lemma 2.2, whose proofs are almost the same as those ones in these lemmas.

Lemma 3.1. *Let $T, R \in \mathbb{A}$ with $T$ monic. Then the $\mathbb{A}$-module $[T, R + y]$ becomes an ideal of $\mathcal{O}_K$ if and only if $T | \mathcal{N}(R + y) = R^2 + RG + D_1Q$. In this case the followings hold:*

(i) *If $T = T_1T_2$, then $[T, R + y] = [T_1, R + y][T_2, R + y]$*
(ii) *If $0 < \deg T < \deg D_1Q$, then $[T, R + y]$ is not a principal ideal.*

Lemma 3.2. *Let $T, R \in \mathbb{A}$ with $T$ monic, $T | \mathcal{N}(R + y)$ and $\deg \mathcal{N}(R + y) < 2 \deg D_1Q$. Then the ideal $[T, R + y]$ is principal if and only if $T = 1$ or $T = \mathcal{N}(R + y)$.*

### 3.2. Ono invariant of $K$

Define the polynomial $f_{(D_1,D_2)}(\mathbf{x}) = \mathbf{x}^2 + G\mathbf{x} + D_1Q$. Then the *Ono invariant* $\rho_{(D_1,D_2)}$ *of* $K$ is defined to be

$$\rho_{(D_1,D_2)} := \max \left\{ \omega(f_{(D_1,D_2)}(R)) : \deg R < \deg D_1Q \right\}.$$

Theorem 3.3. $h_K \geq \rho_{(D_1,D_2)}$.

*Proof.* By definition, $\rho_{(D_1,D_2)} = \omega(f_{(D_1,D_2)}(R))$ for some $R$ with $\deg R < \deg D_1Q - 1$. Note that $f_{(D_1,D_2)}(R) = R^2 + GR + D_1Q = \mathcal{N}(R + y)$ and $\deg \mathcal{N}(R + y) < 2 \deg D_1Q$ because $\deg G = \deg D_2Q/2 \leq$

$\deg D_1Q/2$. We may assume that $n = \omega(f_{(D_1,D_2)}(R)) \geq 1$. Let $f_{(D_1,D_2)}(R) = Q_1 \cdots Q_n$ be the (monic) irreducible factorization of $f_{(D_1,D_2)}(R)$. Write $\mathfrak{a}_i = [Q_i, R+y]$, which is an ideal of $\mathcal{O}_K$ for $1 \leq i \leq n$. As in the proof of Theorem 2.3, we can show that the ideal classes $(\mathfrak{a}_1), (\mathfrak{a}_1\mathfrak{a}_2), \ldots, (\mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_n)$ are mutually distinct, so that we get the result. □

THEOREM 3.4. *$\rho_{(D_1,D_2)} = 1$ if and only if $h_K = 1$.*

*Proof.* By Theorem 3.3, it is clear that if $h_K = 1$, then $\rho_{(D_1,D_2)} = 1$. Now we will show that if $\rho_{(D_1,D_2)} = 1$, then $h_K = 1$. Suppose that $h_K \geq 2$. Let $\mathfrak{p}$ be a non-principal ideal having the smallest $\deg N\mathfrak{p}$. Then we can see that $\mathfrak{p}$ is a prime ideal and $N\mathfrak{p} = P$ is a monic irreducible with $\deg P \leq \deg G - 1 \leq \frac{\deg D_1Q}{2} - 1$. We may write $\mathfrak{p} = [P, R+y]$ for some $R \in \mathbb{A}$ with $\deg R < \deg P$ and $P|\mathcal{N}(R+y)$. Since $\deg R < \deg P \leq \frac{\deg D_1Q}{2} - 1$, we have $\deg \mathcal{N}(R+y) = \deg(R^2+GR+D_1Q) < 2\deg D_1Q$. Since $\rho_{(D_1,D_2)} = 1$, $f_{(D_1,D_2)}(R) = \mathcal{N}(R+y) = R^2+GR+D_1Q$ must be an irreducible. Hence, $P = \mathcal{N}(R+y)$ and $\mathfrak{p} = [P, R+y] = (R+y)$ is principal, which is a contradiction. □

THEOREM 3.5. *$\rho_{(D_1,D_2)} = 2$ if and only if $h_K = 2$.*

*Proof.* If $h_K = 2$, then $\rho_{(D_1,D_2)} \leq h_K = 2$, so $\rho_K = 1$ or $2$. But, by Theorem 3.4, we have $\rho_K = 2$. Conversely, we assume that $\rho_{(D_1,D_2)} = 2$. Then $h_K \geq \rho_{(D_1,D_2)} = 2$. Let $\mathfrak{p}, \mathfrak{p}'$ be non-principal prime ideals with $N\mathfrak{p} = P, N\mathfrak{q} = P'$ and $\deg P, \deg P' \leq \frac{\deg D_1Q}{2} - 1$. Then we can show that $P, P'$ are monic irreducibles and $\mathfrak{p}\mathfrak{p}'$ is principal. Now, for the rest of proof, we follow the argument in the proof of Theorem 2.6 to get the result. □

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